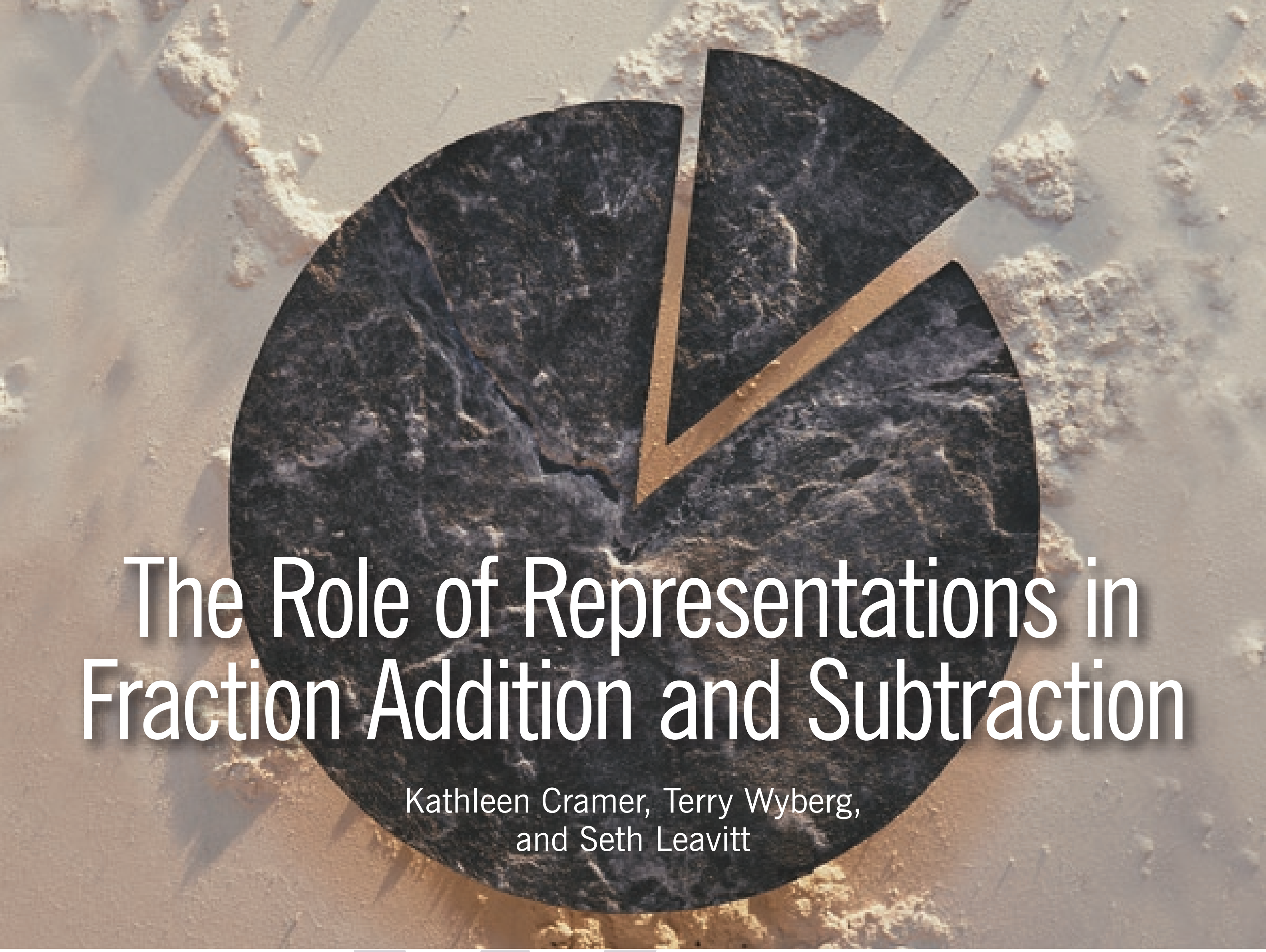

The Role of Representations in Fraction Addition and Subtraction

Kathleen Cramer, Terry Wyberg, and Seth Leavitt

Why do so many middle school students find fraction addition and subtraction difficult despite the fact that they have studied this topic since third or fourth grade? The Rational Number Project (RNP) (Cramer and Henry 2002; Cramer, Post, and delMas 2002) with support from the National Science Foundation is currently engaged in a teaching experiment with sixth graders from a large urban district in the Midwest to address this question.

Representation plays an important role when students are learning about fractions. "Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others" (NCTM 2000, p. 67). Concrete models are critical forms of representation and are needed to support students' understanding of, and operations with, fractions. Other important representations include pictures, contexts, students' language, and symbols. Translating among all these representations makes ideas meaningful to students.

We have found the fraction circle model to be the most powerful concrete representation for fractions. The circle model helps build understanding of the part-whole model for fractions and the meaning of the relative size of fractions. Fraction circles are also a powerful model for fraction addition and subtraction. In this article, we reflect on what we believe is involved in developing a deep understanding of fraction addition and subtraction

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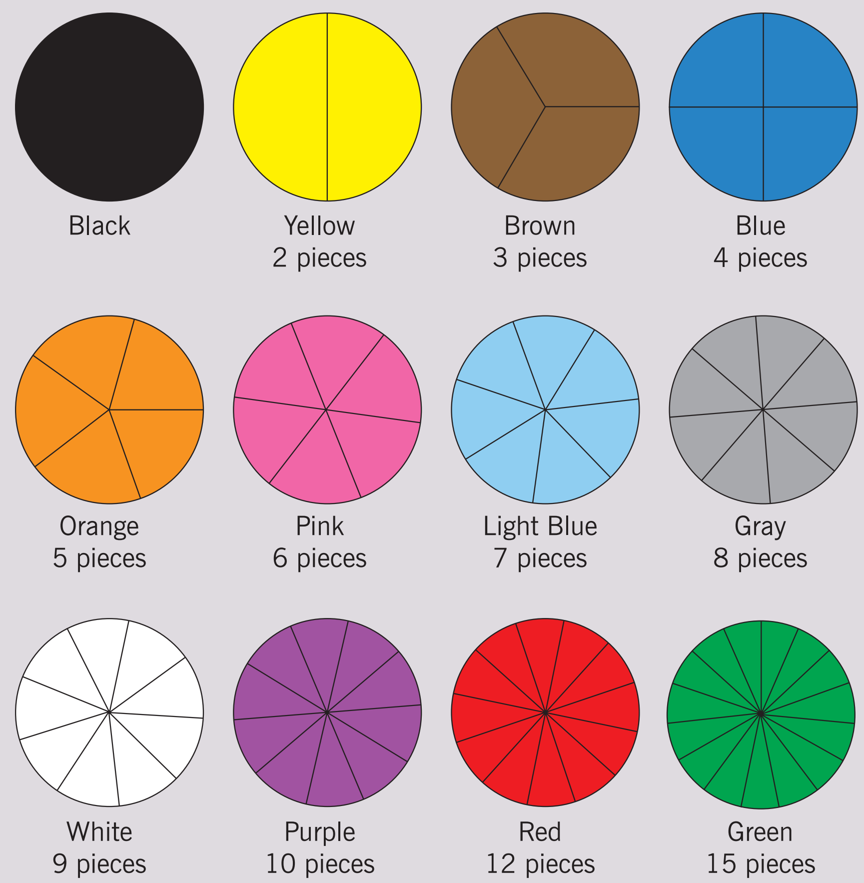
based on our previous work as well as our current teaching experiment. In our current work with two classrooms of sixth graders, students used fraction circles to review ideas involving modeling fractions, comparing fractions, finding equivalent fractions, and acting out fraction addition and subtraction concretely without connecting to symbols. Students then worked through a series of eight additional lessons to develop the meaning of common denominators for adding and subtracting fractions, with the goal of being able to add and subtract fractions symbolically. In the rest of this article, we share our reflections on the role that representations and connections among representations play in helping students understand fraction addition and subtraction procedures.

In our early teaching experiments, we used a variety of different models (fraction circles, chips, paper folding, Cuisenaire rods) to build meaning for fractions. The RNP has found that the fraction circle model is the most effective representation for building mental images for fractions. These mental images support the ability to judge the relative size of fractions, which is an essential skill in fraction understanding (Cramer and Henry 2002). **Figure 1** shows the fraction circles that students used in the RNP teaching experiments. We continue to use fraction circles in the current study. Fraction circles support students' construction of the inverse relationship between the size of the denominator and the size of the fractional piece. Students readily see that the more a circle is partitioned into equal-sized parts, the smaller each part becomes. When asked to pick which fraction is larger, 3/10 or 3/20 (assuming the same-sized unit), one student said, "Well, 10 has like, they are bigger pieces than 20 because there are less pieces." When asked to compare 5/6 with 4/5, the student said, "[The fraction] 4/5 is smaller because they are both one away from a whole, but fifths are bigger pieces than sixths so there is a bigger piece missing." Notice that the student's language is strongly related to the fraction circle model; she is picturing pieces of a circle and seeing that both fractions are one fraction piece away from the whole. Her mental images are strong enough to allow her to be able to visualize that the size of each piece differs.

Fig. 1 Fraction circles



The ability to compare fractions with benchmarks will support students' skill in finding reasonable estimates to fraction addition and subtraction problems. Students should be able to determine if a fraction is close to 0, greater than or less than 1/2, or close to 1. They should also understand what constitutes one whole. In our lessons, we consistently posed questions about the ordering of fractions by relative size to assess students' understanding of the numbers they were to operate on. Consider this task:

> Order from smallest to largest this set of fractions: 3/4, 1/10, 5/12, 3/5, 14/15.

After correctly ordering the fractions from smallest to largest, 1/10, 5/12, 3/5, 3/4, 14/15, the student reasoned as follows:

> 1/10 is the smallest because it is farthest from the whole. I know 5/12 isn't half of twelve yet, so it is under 1/2. Then 3/5. It is two away from a whole number. Then 3/4 and 14/15 because both are missing one piece but a fourth is a lot bigger than a fifteenth, so it is missing a bigger piece.

The student understood that 1/10 was quite small. He recognized that 5/12 is less than 1/2 and probably also knew that 3/5 was greater than 1/2. He compared two fractions close to 1 by accurately picturing the size of the missing piece. Based on our experiences, we concluded:

Before operating with fractions, students need to understand what a fraction means. This involves understanding the part-whole model for fractions and the ability to judge the relative size of a fraction.

The ability to order fractions plays an important role in students' estimation skills. The following question was posed to the sixth graders within individual interviews after two weeks of instruction:

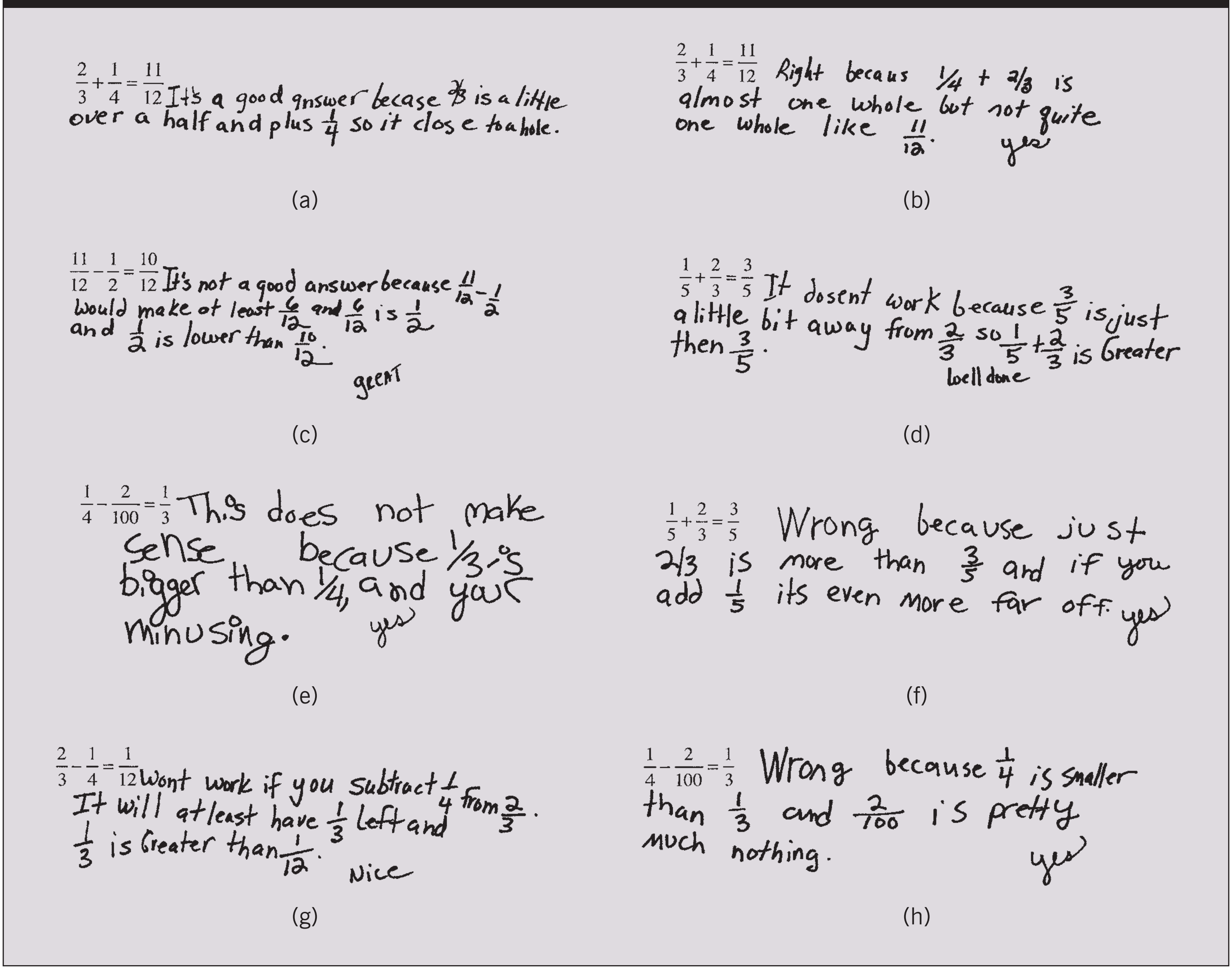
> Marty made two types of cookies. He used 1/5 cup of flour for one recipe and 2/3 cup of flour for the other. How much flour did he use in all? Is it greater than 1/2 cup or less than 1/2 cup? Is the amount greater than 1 cup or less than 1 cup?

(Although 1/5 cup is not as realistic a measure as 2/3 cup of flour, we used that value to encourage students to apply ordering ideas related to comparing unit fractions. As you see in the students' answers, knowing that 1/5 is less than 1/3 gave students the information needed to conclude that the total amount of flour must be less than 1 cup.)

Here are some student responses, with additional questions placed in brackets:

> He used more than 1/2 cup because 2/3 is more than 1/2. He used less than 1 cup. Because 1/5 is less than 1/2. [Why would that be enough to know that the answer is less than 1?] Because 1/5 is 1 out of 5 and so it's smaller pieces. [What would you

Fig. 2 Sixth-grade students' thinking on homework assignments about estimation



need to add to 2/3 to equal 1 cup?]
1/3; 1/3 is bigger than 1/5.

It is already more than 1/2 because 2/3 is greater than 1/2. It's going to be less than 1 cup because 1/5 is not equal to 1/3 so it won't cover that space.

More than 1/2 cup because 2/3 is already bigger than 1/2 so plus 1/5 would be over 1/2. It's less than 1 cup. Because 2/3, 1/5 doesn't fill up the missing piece in 2/3. It just fills up a little pieces of it so it wouldn't equal a whole.

Students used 1/2 as a benchmark, compared unit fractions, and determined amounts to add to 2/3 to make a whole. Students were able to do this because they constructed strong mental images for fractions. Based on student responses to estimation tasks like this one, we concluded:

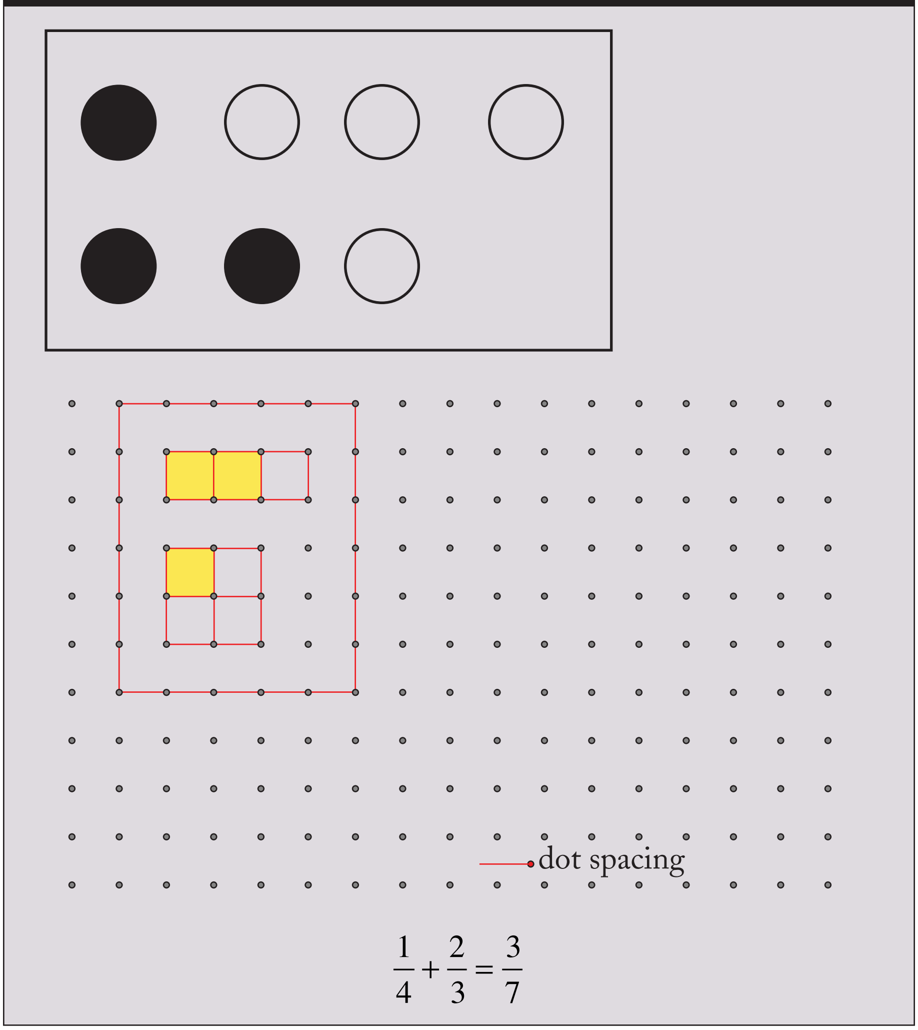
Estimation and visualization are important. These abilities will help students monitor their work when finding exact answers.

Figure 2 shows examples of sixth graders' thinking on estimation tasks drawn from homework assigned before students were asked to find exact answers to fraction addition and subtraction problems. Students were asked if the solution given in each problem was a reasonable estimate. Notice that students coordinated several ordering ideas to judge if the answers to addition and subtraction tasks were reasonable. Their work with fraction circles helped them judge reasonableness, because they were able to visualize the relative sizes of the fractions and combine them by mentally adding on or taking away. Students were able to visualize almost any fraction, regardless of whether they had actually seen that fraction with the fraction circle model. Note the student's response **in figure 2h**. Although there was no fraction circle divided into hundredths, he knew that hundredths were very small pieces, and he only had two of them, so he explained that "2/100 is pretty much nothing."

We were impressed with the depth of understanding that this group of sixth graders showed regarding fraction meaning, their ability to judge the relative sizes of fractions using multiple strategies, and their use of order ideas to estimate fraction addition and subtraction. We strongly believe that the fraction circle model gave students the mental imagery for fractions that supported their understanding. This understanding provided a strong foundation on which to build the concept of a common denominator for adding and subtracting fractions.

In planning the lessons for this latest teaching experiment, we considered which representation might best demonstrate the common denominator procedure for adding and subtracting fractions. We looked at other models, such as pattern blocks, chips, and dot paper, which had been used by other NSF-supported curricula. We looked closely at the positives and negatives of these models using data we collected during another fraction study. We found that students had difficulty simply representing fractions with the pattern blocks and were unable to use them to add fractions.

Fig. 3 Students' errors with chip and dot paper models



Students' reaction to the chip and dot paper model was interesting in that they used it to reinforce the incorrect strategy of adding numerators and denominators (see **fig. 3**). Notice that the students did not use a common unit to show each fraction. Instead they used a unit equal to the number of parts in each denominator. They added the number of parts shaded and the total number of parts to calculate 3/7 as the answer.

Students who tried to model addition with chips or dot paper did not realize the need to express both fractions as equivalent using the same number of chips in the unit or the same number of square units in the dot paper model. In other words, they did not realize the need to express both fractions as equivalent with a common denominator. When considering what the students did with the chips and dot paper to try to model fraction addition, we realized that nothing is obvious in the model itself to show the need for finding common denominators. To add two fractions accurately using chips and dot paper, students need to know already that they have to find a common denominator and realize how the denominator relates to the final answer.

We believe that students would understand common denominators if they constructed for themselves the need to find common denominators. The fraction circle model made this need more obvious than other models we examined. Therefore, in our current teaching experiment, we began fraction addition and subtraction with fraction circles and estimation. We asked students to show 1/2 (a yellow piece) plus 1/6 (a pink piece) on their fraction circle unit (the black circle). We asked, "How much of the whole circle was covered?" The discussion led to an estimate that the answer was obviously greater than 1/2 but less than 1. Students had difficulty naming the exact amount shown on the black fraction circle. One student suggested that to get the exact answer, you need to have all the pieces the color of the smallest piece. We concretely exchanged the yellow piece (1/2) for 3 pinks (3/6). Students' previous work with equivalence ideas based on the fraction circle model supported this step of finding common denominators. Students easily saw that 4/6 of the whole circle was covered, which became an "A ha!" moment for some. At this point, it was tempting to jump right into the procedure but we held off. Finding a common denominator is critical to understanding how to add and subtract with symbols. To truly grasp this idea, a majority of students need extended periods of time with the fraction circles, determining for themselves the need to change the different pieces on the circle to those that are of the same color. Students solved many problems using the fraction circles to add and subtract before they made the connection between exchanging circle pieces and finding common denominators symbolically. We concluded:

Students need to experience acting out addition and subtraction concretely with an appropriate model before operating with symbols.

Moving from a concrete representation to a symbolic one is not as simple as it seems. Written language and verbal language play important roles in helping students translate from concrete to symbolic representation. Students need time to describe what they do with their concrete models before they are able to use symbols to meaningfully record their work. **Figure 4** illustrates how sixth-grade students use written language to describe what they did with the fraction circles. The idea of exchanging equivalent fractions to find common denominators is shown in these examples. For some students, recording with symbols meant using numbers. For others, it meant writing in words.

Students had many opportunities to connect their actions with the fraction circles to pictures of fraction circles to symbols. Students' written explanations were often messy, which sometimes led to errors. After much discussion about, and guidance from, *Principles and Standards for School Mathematics* (NCTM 2000), we concluded that students' own recording systems would be more meaningful than a teacher directed one: "It is important to encourage students to represent their ideas in ways that make sense to them, even if their first representations are not conventional ones" (NCTM 2000, p. 67). For example, many sixth-grade students used arrows to show the change from the given fraction to an equivalent form. This seems to be a reasonable way to show what is being done with the procedure. **Figure 5** shows a few examples of how students recorded their fraction addition and subtraction. We concluded:

Making connections between concrete actions and symbols is an important part of understanding. Students should be encouraged to find their own way of recording with symbols.

At some point, sixth-grade students need experiences with fractions (denominators less than 20) at the symbolic level. Students need to be able *to imagine* fraction circles to help solve problems, but they must actually calculate with symbols (numbers). We found that even though almost all students knew what to do to add and subtract fractions symbolically, their ability to do so easily without the concrete model depended on how quickly they were able to recall their facts. Students who still counted on their fingers were at a disadvantage

Fig. 4 Connecting actions with circles and pictures to symbols

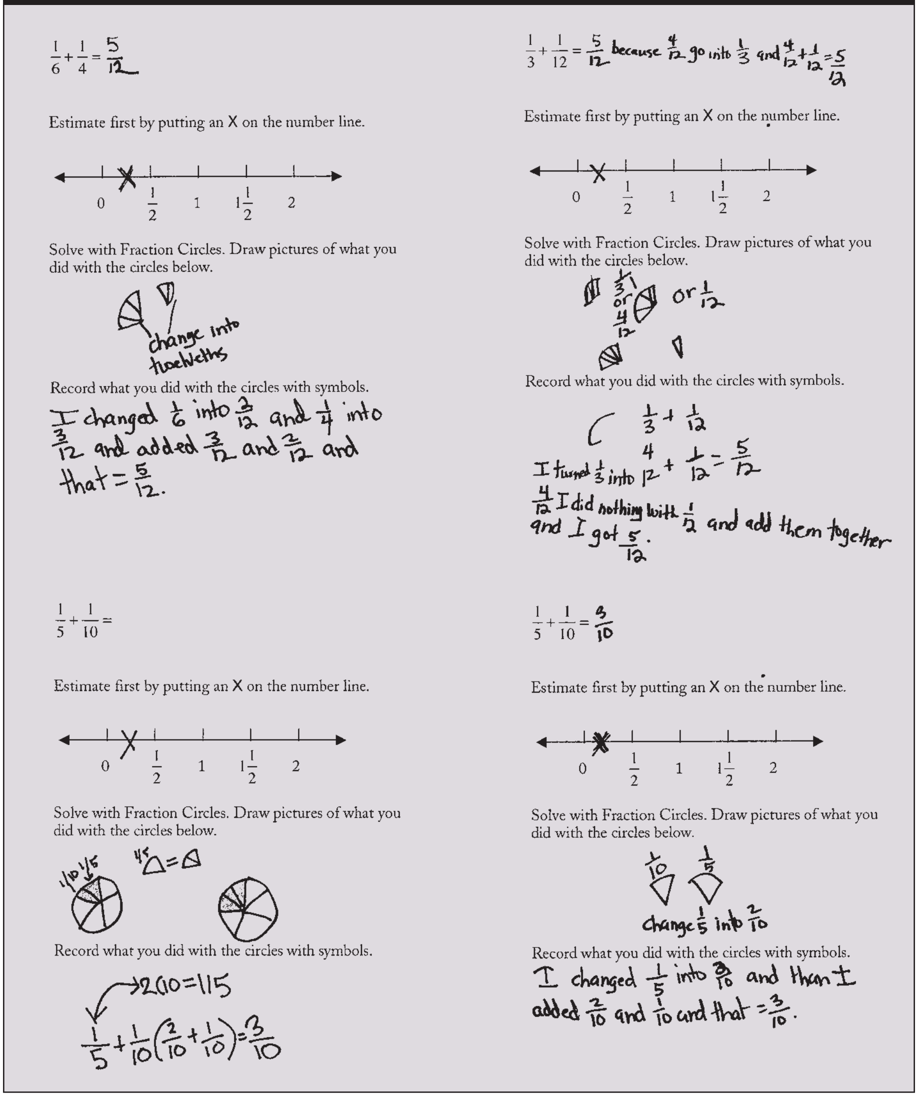


Fig. 5 Students' recording systems

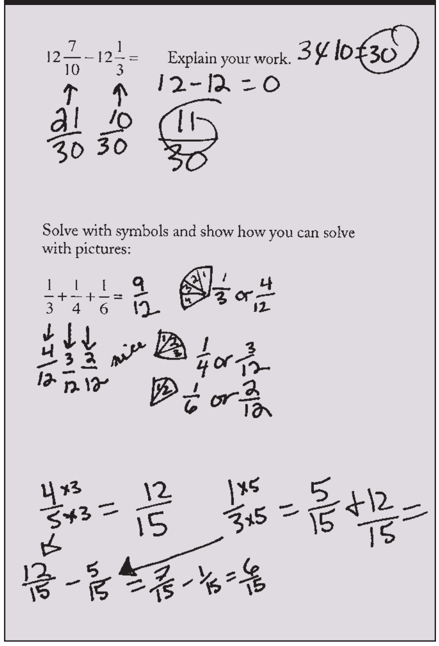
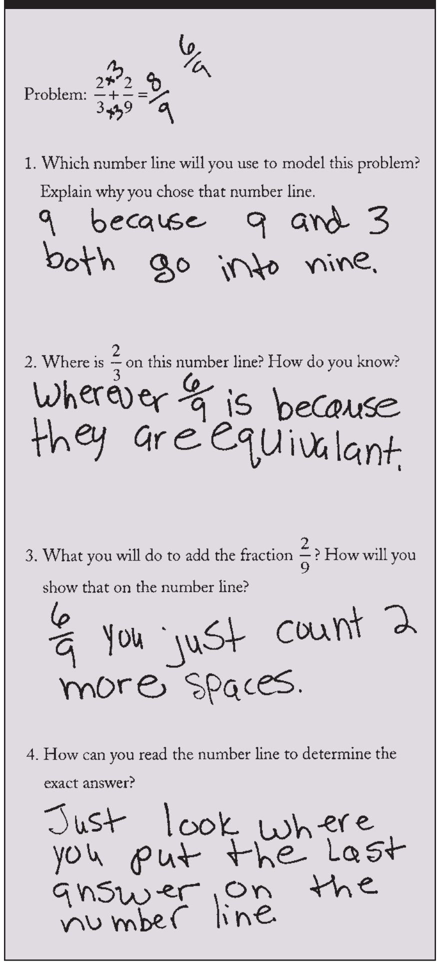


Fig. 6 Student's description of using a number line to add two fractions



when working only with symbols. We found the following:

Students need easy recall of their multiplication and division facts.

Our past experience observing students using chips and dot paper to add and subtract fractions and our new work employing a number line to add and subtract fractions led us to consider the power of connecting their newly learned procedure for adding and subtracting fractions to a different model as a way to reinforce the procedure. As mentioned earlier, some of the models found in current textbooks rely on students being able to understand the need for a common denominator to be able to use these models. For example, to add 3/4 + 1/5 with chips, the student needs to understand that he or she has to represent each fraction using the same number of chips. Determining that number is equivalent to finding the common denominator. Modeling each fraction with that number of chips corresponds to finding equivalent fractions with the common denominator.

In our current work, we used the number line model for addition and subtraction after students learned to find a common denominator. We

found that although it is an effective model for reinforcing the procedure for adding and subtracting fractions, it is less so for the introduction of the concept of common denominators. Once the idea of a common denominator was developed with fraction circles, the number line was introduced to reinforce it. Students used their understanding of common denominators to make sense of the number line as a model for adding and subtracting fractions. Consider the student's work in **figure 6**. Students were given a page of eight number lines partitioned differently: halves, thirds, fourths, sixths, eighths, ninths, tenths, and twelfths. They were asked to show how to add 2/3 and 2/9 on a number line. The decision as to which number line to use to add the two fractions corresponds to finding the common denominator. Identifying each fraction as an equivalent amount on the number line is supported by a student's ability to find equivalent fractions symbolically. Finding the total amount on the number line is the final step involving adding fractions now that each are shown using common denominators. We believe the following:

Connecting the procedure to a new representation may be an effective strategy to reinforce the procedure.

SUMMARY

The concrete models we choose to help students build meaning for fractions and operations are important. In our work, we have determined that the fraction circle model supports students' understanding of the part-whole model for fractions and provides students with mental representations that enable them to judge the relative size of fractions. Students are able to estimate answers to fraction addition and subtraction tasks and judge reasonableness of answers to fraction operation tasks because they have strong mental representations for these numbers and are able to manipulate them mentally. In our latest teaching experiment, we developed lessons to help students build meaning for finding exact answers to fraction addition and subtraction problems using common denominators. We found that the fraction circles vividly demonstrate the need for finding common denominators when adding and subtracting fractions and that the fraction circles show the steps to exchanging given fractions with equivalent ones with common denominators. We will continue to study how fraction circles and other models contribute to students' mastery of this difficult procedure.

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